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# EXACT SOLUTION OF THE PROBLEM OF NONSTATIONARY HEAT CONDUCTION FOR TWO SEMISPACES IN NONIDEAL CONTACT

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*An exact solution of the nonideal-contact problem of nonstationary heat conduction for two semispaces with constant initial temperatures has been obtained. It is shown that the problem at hand is similar in thermal action to the third boundary-value problem for a semispace bordering on a medium of constant temperature that has a certain value.*

Let a contact surface (plane $x = 0$) separate two semispaces with different thermophysical properties and initial temperatures $T_{10}$ = const and $T_{20}$ = const ($T_{10} \neq T_{20}$). The distribution of the temperatures $T_1(x, t)$ and $T_2(x, t)$ in the case of a nonideal contact at any instant of time is described by the following equations by heat conduction and initial and boundary conditions:

$$\frac{\partial T_1}{\partial t} = a_1 \frac{\partial^2 T_1}{\partial x^2}, \quad x < 0, \quad t > 0, \qquad \frac{\partial T_2}{\partial t} = a_2 \frac{\partial^2 T_2}{\partial x^2}, \quad x > 0, \quad t > 0; \tag{1}$$

$$T_1(x, 0) = T_1(-\infty, t) = T_{10} < \infty, \quad x < 0, \qquad T_2(x, 0) = T_2(+\infty, t) = T_{20} < \infty, \quad x > 0; \tag{2}$$

$$\lambda_1 \left.\frac{\partial T_1}{\partial x}\right|_{x=-0} = \lambda_2 \left.\frac{\partial T_2}{\partial x}\right|_{x=+0} = \alpha \left[T_2(+0, t) - T_1(-0, t)\right]. \tag{3}$$

In order to find the solution of problem (1)–(3), we introduce two unknown adjoint functions of time equal to the temperatures at the zone of contact: $\varphi(t) = T_1(-0, t)$ and $\psi(t) = T_2(+0, t)$. Note that in solving ideal-contact problems, usually one unknown function of time is introduced, identified with the heat flux at $x = 0$ [1]. We will apply the Laplace transform in time to (1)–(3). Then, integrating the resulting two ordinary differential equations with corresponding boundary conditions, we find the images of temperature distributions in each semispace that depend on the images $\varphi(s)$ and $\psi(s)$ (precisely, on their difference): $T_1(x, s) = f[x, \varphi(s) - \psi(s)]$ and $T_2(x, s) = g[x, \varphi(s) - \psi(s)]$, where $f$ and $g$ are certain functions. Equating the left sides taken at $x = 0$ to the images of the functions introduced above, we obtain a system of two linear algebraic equations: $\varphi(s) = f[0, \varphi(s) - \psi(s)]$ and $\psi(s) = g[0, \varphi(s) - \psi(s)]$. Having solved it for $\varphi(s)$ and $\psi(s)$ and substituted the latter functions into $f$ and $g$, we find the images of the temperature profiles. The inverse transforms that represent the exact solutions of problem (1)–(3) have the form

$$T_1(x, t) = T_{10} - \frac{T_{10} - T_{20}}{1 + K_{\varepsilon_1}} \left[\operatorname{erfc}\left(-\frac{x}{2\sqrt{a_1 t}}\right) - \exp\left(-\frac{bx}{\sqrt{a_1}} + b^2 t\right) \operatorname{erfc}\left(b\sqrt{t} - \frac{x}{2\sqrt{a_1 t}}\right)\right], \quad x < 0; \tag{4}$$

$$T_2(x, t) = T_{20} + \frac{T_{10} - T_{20}}{1 + K_{\varepsilon_2}} \left[\operatorname{erfc}\left(\frac{x}{2\sqrt{a_2 t}}\right) - \exp\left(\frac{bx}{\sqrt{a_2}} + b^2 t\right) \operatorname{erfc}\left(b\sqrt{t} + \frac{x}{2\sqrt{a_2 t}}\right)\right], \quad x > 0, \tag{5}$$

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where

$$K_{\varepsilon_1} = \sqrt{\frac{\lambda_1 c_1 \gamma_1}{\lambda_2 c_2 \gamma_2}} = \frac{\varepsilon_1}{\varepsilon_2}\,; \quad K_{\varepsilon_2} = \frac{1}{K_{\varepsilon_1}}\,; \quad b = H_1 \sqrt{a_1} + H_2 \sqrt{a_2} = \alpha \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right); \quad H = \frac{\alpha}{\lambda}\,.$$

Solutions (4), (5) can be represented in a relative dimensionless form as

$$\theta_1 = \frac{T_1(x, t) - T_{10}}{T_{10} - T_{20}} = -\frac{1}{1 + K_{\varepsilon_1}} \left\{ \operatorname{erfc}\left(\frac{1}{2\sqrt{\mathrm{Fo}_1}}\right) - \exp\,[\mathrm{Bi}_1^* + (N^*)^2]\,\operatorname{erfc}\left(\frac{1}{2\sqrt{\mathrm{Fo}_1}} + N^*\right)\right\}, \tag{6}$$

$$\theta_2 = \frac{T_2(x, t) - T_{20}}{T_{10} - T_{20}} = \frac{1}{1 + K_{\varepsilon_2}} \left\{ \operatorname{erfc}\left(\frac{1}{2\sqrt{\mathrm{Fo}_2}}\right) - \exp\,[\mathrm{Bi}_2^* + (N^*)^2]\,\operatorname{erfc}\left(\frac{1}{2\sqrt{\mathrm{Fo}_2}} + N^*\right)\right\}. \tag{7}$$

In Eqs. (6) and (7), the effective generalized variables that account for the thermal mutual effect of the semispaces are introduced. They are expressed in terms of ordinary Fourier numbers, numbers of homochronicity, Biot number, and thermal activities in the following form: $\mathrm{Bi}_1^* = \mathrm{Bi}_1(1 + K_{\varepsilon_1})$ and $\mathrm{Bi}_2^* = \mathrm{Bi}_2(1 + K_{\varepsilon_2})$, $N^* = N + N_2 = \mathrm{Bi}_1^*\sqrt{\mathrm{Fo}_1} = \mathrm{Bi}_2^*\sqrt{\mathrm{Fo}_2} = \alpha\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\sqrt{t}$. The latter system of equalities means that the effective homochronicity number is equal to the sum of the homochronicity numbers of the first and second semispaces. In turn, the ordinary generalized variables are defined by the equalities $\mathrm{Bi}_1 = \frac{\alpha x}{\lambda_1}$, $\mathrm{Bi}_2 = \frac{\alpha x}{\lambda_2}$, $\mathrm{Fo}_1 = \frac{a_1 t}{x^2}$, $\mathrm{Fo}_2 = \frac{a_2 t}{x^2}$, $N_1 = \mathrm{Bi}_1\sqrt{\mathrm{Fo}_1}$, and $N_2 = \mathrm{Bi}_2\sqrt{\mathrm{Fo}_2}$, with the following relationships existing in the problem considered: $\mathrm{Bi}_1\sqrt{\mathrm{Fo}_1} + \mathrm{Bi}_2\sqrt{\mathrm{Fo}_2} = \mathrm{Bi}_1\sqrt{\mathrm{Fo}_1}(K_{\varepsilon_1} + 1) = \mathrm{Bi}_2\sqrt{\mathrm{Fo}_2}(K_{\varepsilon_2} + 1)$.

We will analyze the solution obtained. In the particular case where $N^* \to \infty$ (for example, when the heat-transfer coefficient is infinitely large or one of the quantities of thermal activities is infinitely small), using the expansion of the function erfc $(\cdot)$ at large arguments in (6) and (7), we may establish that the terms in the curly brackets involving $N^*$ tend to $1/\sqrt{\pi}N^*$, and they can be neglected. The resulting expressions $\theta_1 = -\frac{1}{1 + K_{\varepsilon_1}}\operatorname{erfc}\left(\frac{1}{2\sqrt{\mathrm{Fo}_1}}\right)$ and $\theta_2 = \frac{1}{1 + K_{\varepsilon_2}}\operatorname{erfc}\left(\frac{1}{2\sqrt{\mathrm{Fo}_2}}\right)$ are nothing but the solution of the problem of an ideal thermal contact. In the other limiting case with $\mathrm{Bi}^* \to 0$ and $N^* \to 0$ there is no heat transfer, and everywhere the temperatures are equal to their initial values. By comparing we may find out that solution (6) in terms of the variables $\mathrm{Fo}_1$ and $\mathrm{Bi}_1^*$ coincides to within the factor $1/(1 + K_{\varepsilon_1})$ with the solution in the variables $\mathrm{Fo}_1$ and $\mathrm{Bi}_1$ (we will denote it by $\Theta_1(\mathrm{Fo}_1, \mathrm{Bi}_1)$) of the third boundary-value problem (see, e.g., [2], p. 184) for semispace 1 with the initial temperature $T_{10}$, provided that the temperature of the adjoining semispace 2 all the time remains constant and equal to $T_{20}$. This means that the temperature field $\theta_1(\mathrm{Fo}_1, \mathrm{Bi}_1^*)$ can be obtained from the field $\Theta_1(\mathrm{Fo}_1, \mathrm{Bi}_1)$ by the method of compression of the scale of temperature $\Theta_1$ and extension of the scale of the variable $\mathrm{Bi}_1$ by a factor of $1 + K_{\varepsilon_1}$, i.e., $\theta_1$ and $\Theta_1$ are similar. When $K_{\varepsilon_1} << 1$, it follows from (12) and (13) that $\theta_1$ and $\Theta_1$ coincide, i.e., $\mathrm{Bi}_1^* = \mathrm{Bi}_1$, $\mathrm{Bi}_2^* = \infty$, $N^* = N_1$, $\theta_2 = 0$, and $\theta_1 = \Theta_1$. Similar reasonings are also applicable to $\theta_2$ found from (7), but here semispaces 1 and 2 change their roles and the subscripts change their places. Thus, by the thermal effect the problems compared are similar to each other.

## NOTATION

$a = \lambda/(c\gamma)$, thermal diffusivity; $b$, thermal mutual effect of semispaces depending on the degree of nonideality of the contact; Bi and $\mathrm{Bi}^*$, Biot number and its effective value; $c$, isochoric heat conductivity; Fo, Fourier number; $H$, reduced heat-transfer coefficient; $K_{\varepsilon_1}$ and $K_{\varepsilon_2}$, criteria of the thermal activity of the first semispace relative to the second semispace and the other way around; $N$ and $N^*$, the homochronicity number and its effective value; $s$, complex variable in the Laplace transform; $T$, temperature; $t$, time; $x$, coordinate; $\alpha$, heat-transfer coefficient; $\gamma$, density; $\varepsilon$, thermal activity; $\theta$, relative temperature; $\lambda$, thermal conductivity; $\varphi$ and $\psi$, adjoint functions equal to the values of $T_1$ and $T_2$ at $x = 0$; erf $(\cdot)$, probability integral; erfc $(\cdot) = 1 -$ erfc $(\cdot)$. Subscripts: 1 and 2, the first and second semispaces, 10 and 20, their initial values.

## REFERENCES


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